

# Singular connection and Riemann theta function

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## Abstract

We prove the Chern-Weil formula for  $SU(n+1)$ -singular connections over the complement of an embedded oriented surface in smooth four manifolds. The expression of the representation of a number as a sum of nonvanishing squares is given in terms of the representations of a number as a sum of squares. Using the number theory result, we study the irreducible  $SU(n+1)$ -representations of the fundamental group of the complement of an embedded oriented surface in smooth four manifolds.

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## 1 Introduction

We study  $SU(n+1)$ -singular connections over  $X \setminus \Sigma$  in this paper, where  $X$  is a smooth closed oriented 4-manifold and  $\Sigma$  is a closed embedded surface. In [6], S. Wang first started to understand the topological information from singular connections. Later, Kronheimer and Mrowka [3] studied the Donaldson invariants under the change of  $SU(2)$ -singular connections. The paper [3] turns out to be a crucial step for analyzing the structure of the Donaldson invariants and for recent development in the Seiberg-Witten theory.

In §2, we describe the  $SU(n+1)$ -singular connection space over  $X \setminus \Sigma$ . The  $SU(n+1)$ -singular Chern-Weil formula is given in Proposition 2.2. In order to study the irreducible  $SU(n+1)$ -representations, we need to study the sum of nonvanishing squares. This is one of well-known number theory problems. Jacobi initially studied the representation  $r_k(n)$  of a number  $n$  as a sum of  $k$ -squares via Riemann theta function as a generating function. For a topological reason, we would like to understand the representation  $\mathcal{R}_k(n)$  of a number  $n$  as a sum of nonvanishing  $k$ -squares. In general it is difficult to calculate both  $r_k(n)$  and  $\mathcal{R}_k(n)$ . We prove a nice relation between  $r_k(n)$  and  $\mathcal{R}_k(n)$  in Proposition 3.1. Up to the author's knowledge, it has not known before how to express the number  $\mathcal{R}_k(n)$  (see [2]). Similar relation for the representation of a number by a quadratic form is also obtained.

In the last section, we use those number theory criterions to study  $SU(n+1)$ -singular flat connections. This gives an interesting interaction between  $\mathcal{R}_n(N)$  in the number theory and the irreducible  $SU(n+1)$ -representations of  $\pi_1(X \setminus \Sigma)$  in topology (see Proposition 4.2).

## 2 $SU(n+1)$ singular connection and Chern-Weil formula

### (i) Singular connections over $X \setminus \Sigma$

Let  $X$  be a smooth closed oriented 4-manifold and let  $\Sigma$  be a closed embedded surface. We will assume that both  $X$  and  $\Sigma$  are connected, and  $\Sigma$  to be orientable or oriented for simplifying our discussions. Denote  $tN$  be a closed tubular neighborhood of  $\Sigma$ . Identify  $tN$  diffeomorphically to the unit disk bundle of the normal bundle. Let  $Y$  be the boundary of  $tN$ , which has the circle bundle structure over  $\Sigma$  by this diffeomorphism. Let  $i\eta$  be a connection 1-form for the circle bundle, so  $\eta$  is an  $S^1$ -invariant 1-form on  $Y$  which coincides with the 1-form  $d\theta$  on each circle fiber. Using  $(r, \theta)$  polar coordinates in some local trivialization of the disk bundle, we have that  $dr \wedge d\theta$  is the correct orientation for the normal plane. By radial projection,  $\eta$  can be extended to  $tN \setminus \Sigma$ .

We will work on the structure group  $SU(n+1)$  for  $n \geq 1$ . So a connection  $A$  on  $X \setminus \Sigma$  which is represented on each normal plane to  $\Sigma$  by a connection matrix looks like

$$i \begin{pmatrix} \alpha_0 & & & \\ & \alpha_1 & & \\ & & \ddots & \\ & & & \alpha_n \end{pmatrix} d\theta + (\text{lower terms}), \quad \sum_{i=0}^n \alpha_i \equiv 0 \pmod{1}. \quad (2.1)$$

The size of the connection matrix is  $o(r^{-1})$ , so  $A$  is singular along the surface  $\Sigma$ .

For every  $SU(n+1)$ -singular connection, one can associate with holonomy as in [4, 6]. Let  $P \rightarrow X \setminus \Sigma$  be a vector bundle with structure group  $SU(n+1)$  for  $n \geq 1$ . To define the holonomy around  $\Sigma$  of a connection  $A$  on  $P$ , for any point  $\sigma \in \Sigma$  and real number  $0 < r < 1$ , let  $S_\sigma^1(r)$  be a circle with center  $\sigma$  and radius  $r$  on the normal plane of  $tN$  over  $\sigma$ . An element  $h(A; \sigma, r) \in SU(n+1)$  can be obtained by parallel transport of a frame of  $P$  along  $S_\sigma^1(r)$  via the connection  $A$ . Although  $h(A; \sigma, r)$  depends on the choice of a frame, its conjugacy class  $[h(A; \sigma, r)]$  in  $SU(n+1)$  does not (c.f. [4, 6]). If for all  $\sigma \in \Sigma$ , the limit  $h_A = \lim_{r \rightarrow 0^+} [h(A; \sigma, r)]$  is independent of  $\sigma$  and  $tN$ , we call it the holonomy of  $A$  along  $\Sigma$ .

The holonomy of this connection on the positively oriented small circles of radius  $r$  is approximately

$$\exp 2\pi i \begin{pmatrix} \alpha_0 & & & \\ & \alpha_1 & & \\ & & \ddots & \\ & & & \alpha_n \end{pmatrix}, \quad \sum_{i=0}^n \alpha_i \equiv 0 \pmod{1}. \quad (2.2)$$

Since only the conjugacy class of the holonomy has any invariant meaning, we may suppose that  $\alpha_i$  lies in the interval  $[0, 1)$ , therefore the matrices (2.2) modulo the permutation group  $S_{n+1}$  run through each conjugacy class just once. When  $\alpha_i = 0$

for all  $0 \leq i \leq n$ , the holonomy is trivial, and if the phrase “lower terms” makes sense, we have ordinary connections on  $X$ . Also when  $\alpha_i \in \{0, \frac{1}{n}, \dots, \frac{n-1}{n}\}$  for all  $i$ , the holonomy is in the center of  $SU(n+1)$  ( $n$ -th root of unity), the associated  $SU(n+1)/Z_n$ -bundle has trivial holonomy; and with this twist we can consider these as  $PSU(n+1)$ -connections on  $X$ .

Conjugacy classes in  $SU(n+1)$  can be characterized by parameters  $\alpha_i$  with

$$\alpha = (\alpha_i)_{0 \leq i \leq n} \in [0, 1)^{n+1} / (\alpha_0 + \alpha_1 + \dots + \alpha_n \equiv 0 \pmod{1}).$$

Note that any permutation of  $(\alpha_i)$  gives the same conjugacy class. Hence we can stay on the region for conjugacy classes by making  $\alpha_i$  satisfying the following

$$1 > \alpha_0 \geq \alpha_1 \geq \dots \geq \alpha_n \geq 0, \text{ and } \alpha_0 + \alpha_1 + \dots + \alpha_n \in \{0, 1, \dots, n\}. \quad (2.3)$$

The region of conjugacy classes of  $SU(n+1)$  can be identified with the quotient space  $\{z_i \in S^1, \prod_{i=0}^n z_i = 1\} / S_{n+1}$  of the maximal torus of  $SU(n+1)$  under the Weyl group action. When  $n = 1$  ( $SU(2)$  case),  $1 > \alpha_0 \geq \alpha_1 \geq 0, \alpha_0 + \alpha_1 = 0, \alpha_0 + \alpha_1 = 1$ . The equation  $\alpha_0 + \alpha_1 = 0$  always gives the identity conjugacy class. So  $1 > \alpha_0 \geq \alpha_1 \geq 0$  and  $\alpha_0 + \alpha_1 = 1$  describe the conjugacy classes of  $SU(2)$  as in [3] with  $\alpha'_0 = \alpha_0 + 1/2, \alpha'_1 = \alpha_1 + 1/2$ . For  $\alpha_i = 0 \geq \alpha_{i+1} \geq \dots \geq \alpha_n \geq 0$ , the conjugacy classes can be viewed as conjugacy classes in  $SU(i)$  or  $U(i)$  in  $SU(n+1)$ .

The matrix-valued 1-form given on  $X \setminus \Sigma$  by the expression

$$i \begin{pmatrix} \alpha_0 & & & \\ & \alpha_1 & & \\ & & \ddots & \\ & & & \alpha_n \end{pmatrix} \eta, \quad (2.4)$$

has the asymptotic behavior of (2.1), but is only defined locally. To make an  $SU(n+1)$  connection on  $X \setminus \Sigma$  which has this form near  $\Sigma$ , start with  $SU(n+1)$  bundle  $\overline{P}$  over  $X$  and choose a  $C^\infty$  decomposition of  $\overline{P}$  on  $N$  as

$$\overline{P}|_N = \overline{L}_0 \oplus \overline{L}_1 \oplus \dots \oplus \overline{L}_n,$$

compatible with the hermitian metric. Since we will work on the modeled connection  $A^\alpha$ , the decomposition of  $\overline{P}|_N$  gives a natural model. Although  $\overline{P}|_N$  is trivial, but  $\overline{L}_i$  may not be. We define topological invariants in this situation:

$$\begin{cases} k &= c_2(\overline{P})[X] \\ l_i &= -c_1(\overline{L}_i)[\Sigma], \quad \sum_{i=0}^n l_i = 0. \end{cases} \quad (2.5)$$

Choose any smooth  $SU(n+1)$  connection  $\overline{A}^0$  on  $\overline{P}$  which respects to the decomposition over  $N$ , so we have

$$\overline{A}^0|_N = \begin{pmatrix} b_0 & & & \\ & b_1 & & \\ & & \ddots & \\ & & & b_n \end{pmatrix}, \quad \sum_{i=0}^n b_i = 0, \quad (2.6)$$

where  $b_i$  is a smooth connection in  $\overline{L}_i$ . Let the model connection  $A^\alpha$  on  $P = \overline{P}|_{X \setminus \Sigma}$  be the following:

$$A^\alpha = \overline{A}^0 + i\beta(r) \begin{pmatrix} \alpha_0 & & & \\ & \alpha_1 & & \\ & & \ddots & \\ & & & \alpha_n \end{pmatrix} \eta, \quad (2.7)$$

where  $\beta$  is a smooth cutoff function equal to 1 in  $[0, \frac{3}{8}]$  and equal to 0 for  $r \geq \frac{1}{2}$ . In terms of trivialization compatible with the decomposition, the second term in  $A^\alpha$  is an element of  $\Omega_{N \setminus \Sigma}^1(AdP)$  which can be extended to all of  $X \setminus \Sigma$ . The curvature  $F(A^\alpha)$  extends to a smooth 2-form with values in  $Ad\overline{P}$  on the whole  $X$ , since  $id\eta$  is smooth on  $tN$ ,  $i\eta$  is the pullback to  $tN$  of the curvature form of the circle bundle  $Y$ . It can be thought as a smooth 2-form on the surface  $\Sigma$ .

The connection  $A^\alpha$  in (2.7) defines a connection on  $X \setminus \Sigma$ . The holonomy  $h_{A^\alpha}$  around small linking circles is asymptotically equal to (2.2). We now define an affine space of connections modeled on  $A^\alpha$  by choosing  $p > 2$  and denoting

$$\mathcal{A}_1^{\alpha,p} = \{A^\alpha + a \mid \|a\|_{L^p(X \setminus \Sigma)} + \|\nabla_{A^\alpha} a\|_{L^p(X \setminus \Sigma)} < \infty\}.$$

Similarly a gauge group

$$\mathcal{G}_2^{\alpha,p} = \{g \in AutP \mid \|g\|_{L^p(X \setminus \Sigma)} + \|\nabla_{A^\alpha} g\|_{L^p(X \setminus \Sigma)} + \|\nabla_{A^\alpha}^2 g\|_{L^p(X \setminus \Sigma)} < \infty\}.$$

The  $L^p$  space is defined by using the measure inherited from any smooth measure on  $X$ .

**Proposition 2.1** (i) *The space  $\mathcal{A}_1^{\alpha,p}$  and  $\mathcal{G}_2^{\alpha,p}$  are independent of the choices of  $\overline{A}^0$  and the connection 1-form  $\eta$ .*

(ii) *The space  $\mathcal{G}_2^{\alpha,p}$  is a Banach Lie group which acts smoothly on  $\mathcal{A}_1^{\alpha,p}$  and is independent of  $\alpha$ . The stabilizer of  $A$  is  $Z_n$  or  $H$  ( $Z_n \subset H \subset SU(n+1)$ ) according as  $A$  is irreducible or reducible respectively.*

Proof: The proof is same as in the Proposition 2.4 in [3] (see also Chapter 3 [6]). ■

## (ii) The Chern-Weil formula for $\mathcal{A}_1^{\alpha,p}$

By the same token of the Proposition 3.7 in [3], we have that the equivalence class of the norm  $L_{k,A^\alpha}^p$  is independent of  $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n)$ , so the gauge group  $\mathcal{G}_{k+1}^{\alpha,p} = \mathcal{G}_{k+1}^p$  is independent of  $\alpha$  as a parameter in the definition of the model connection  $A^\alpha$ . So the space  $\mathcal{A}_1^{\alpha,p} = A^\alpha + L_1^p(\Omega(X \setminus \Sigma), AdP)$ , where  $L_1^p(\Omega(X \setminus \Sigma), AdP)$  is a Banach space which is independent of  $\alpha$ , and  $a \in L_1^p(\Omega(X \setminus \Sigma), AdP)$ , the diagonal component  $D(a)$  of  $a$  is in  $L_1^p$ , and  $(a - D(a))$  is in  $L_1^p$ ,  $r^{-1}(a - D(a))$  is in  $L^p$ .

**Proposition 2.2** *For all  $A \in \mathcal{A}_1^{\alpha,p}$ , the following Chern-Weil formula holds.*

$$\frac{1}{8\pi^2} \int_{X \setminus \Sigma} tr(F_A \wedge F_A) = k + \sum_{i=0}^n \alpha_i l_i - \frac{1}{2} \left( \sum_{i=0}^n \alpha_i^2 \right) \Sigma \cdot \Sigma. \quad (2.8)$$

Proof: We begin by proving the formula for the model connection  $A^\alpha$  in the simplest case. Let  $A^\alpha$  be globally reducible. So  $\overline{E} = \overline{L}_0 \oplus \overline{L}_1 \cdots \oplus \overline{L}_n$  globally, that  $\overline{b}_i$  is a smooth connection on  $\overline{L}_i$  and that  $A^\alpha$  is reducible as

$$A^\alpha = \begin{pmatrix} b_0 & & & \\ & b_1 & & \\ & & \ddots & \\ & & & b_n \end{pmatrix} \quad \text{with} \quad b_i = \overline{b}_i + i\alpha_i\beta(r)\eta.$$

The cutoff function  $\beta(r)$  is defined in (2.7), and  $i\eta$  is a connection 1-form on the normal circle bundle to  $\Sigma$ . The closed 2-form  $d(i\beta(r)\eta)$  can be extended smoothly across  $\Sigma$ , due to  $\beta(r) = 1$  near  $\Sigma$ . So it is the pullback from  $\Sigma$  of the curvature form  $F(i\eta)$ . Since the second cohomology of the neighborhood is 1-dimensional, so the closed 2-form  $d(i\beta(r)\eta) = F(i\eta)$  represents the Poincaré dual of  $\Sigma$ , denote by  $P.D(\Sigma) = d(i\beta(r)\eta)$ . Hence we have the degree of the normal bundle

$$\int_{\Sigma} -\frac{1}{2\pi i} F(i\eta) = \int_{\Sigma} -\frac{1}{2\pi i} d(i\beta(r)\eta) = \langle P.D(\Sigma), \Sigma \rangle = \Sigma \cdot \Sigma.$$

In de Rham cohomology, we have

$$\begin{aligned} -\frac{1}{2\pi i} d(b_i) &= -\frac{1}{2\pi i} d(\overline{b}_i) + \alpha_i \left( -\frac{1}{2\pi i} d(i\beta(r)\eta) \right) \\ &= c_1(\overline{L}_i) + \alpha_i P.D(\Sigma) \end{aligned} \quad (2.9)$$

So we have the following two identities:

$$\frac{1}{8\pi^2} \text{tr}(F_{A^\alpha} \wedge F_{A^\alpha}) = \frac{1}{8\pi^2} \text{tr}(dA^\alpha \wedge dA^\alpha) = \frac{1}{8\pi^2} \text{tr}\left(\sum_{i=0}^n db_i \wedge db_i\right), \quad (2.10)$$

$$\langle (-\frac{1}{2\pi i} db_i) \wedge (-\frac{1}{2\pi i} db_i), X \rangle = - \langle \frac{1}{4\pi^2} db_i \wedge db_i, X \rangle. \quad (2.11)$$

Therefore by (2.11) and (2.10),

$$\begin{aligned} -\frac{1}{4\pi^2} \langle db_i \wedge db_i, [X] \rangle &= - \langle (c_1(\overline{L}_i) + \alpha_i P.D(\Sigma)) \wedge (c_1(\overline{L}_i) + \alpha_i P.D(\Sigma)), [X] \rangle \\ &= - \langle (c_1(\overline{L}_i))^2, [X] \rangle - 2\alpha_i \langle c_1(\overline{L}_i) \wedge P.D(\Sigma), [X] \rangle \\ &\quad - (\alpha_i)^2 \langle P.D(\Sigma) \wedge P.D(\Sigma), [X] \rangle \\ &= -l_i^2 - 2\alpha_i \langle c_1(\overline{L}_i), \Sigma \rangle - (\alpha_i)^2 \langle P.D(\Sigma), X \cap \Sigma \rangle \\ &= -l_i^2 + 2\alpha_i l_i - \alpha_i^2 \Sigma \cdot \Sigma. \end{aligned} \quad (2.12)$$

Observe that on the Lie algebra  $\mathfrak{su}(n)$  of skew adjoint matrices  $\text{tr}(M^2) = -|M|^2$ . Hence by (2.10) and (2.11) the Chern-Weil formula for the modeled connection  $A^\alpha$  is

$$\frac{1}{8\pi^2} \int_{X \setminus \Sigma} \text{tr} F_{A^\alpha} \wedge F_{A^\alpha} = \frac{1}{2} \sum_{i=0}^n (-l_i^2 + 2\alpha_i l_i - \alpha_i^2 \Sigma \cdot \Sigma). \quad (2.13)$$

Note that  $c_2(\overline{E}) = \sum_{i < j} c_1(\overline{L}_i) \cdot c_1(\overline{L}_j) = \sum_{i < j} l_i l_j$ . Also from  $c_1(\overline{E}) = \sum_{i=0}^n l_i = 0$ , we have

$$0 = \left(\sum_{i=0}^n l_i\right)^2 = \sum_{i=0}^n l_i^2 + 2 \sum_{i < j} l_i l_j,$$

i.e.  $c_2(\overline{E}) = -\frac{1}{2} \sum_{i=0}^n l_i^2$ . By (2.13) we have

$$\frac{1}{8\pi^2} \int_{X \setminus \Sigma} \text{tr} F_{A^\alpha} \wedge F_{A^\alpha} = k + \sum_{i=0}^n \alpha_i \cdot l_i - \frac{1}{2} \left(\sum_{i=0}^n \alpha_i^2\right) (\Sigma \cdot \Sigma). \quad (2.14)$$

Although this calculation is global, it has an interpretation locally on  $tN$ . Let  $Y_\varepsilon \subset tN$  be the 3-manifold circle bundle over  $\Sigma$  given by radius  $r = \varepsilon$ . The Chern-Simons integral is given by the following:

$$cs_\varepsilon(A^\alpha) = \frac{1}{8\pi^2} \int_{Y_\varepsilon} \text{tr}(dA^\alpha \wedge A^\alpha + \frac{2}{3} A^\alpha \wedge A^\alpha \wedge A^\alpha).$$

The integral  $cs_\varepsilon(A^\alpha)$  depends only on the homotopy class of the trivialization of the bundle on  $Y_\varepsilon$  with respect to which the connection matrix  $A^\alpha$  is computed (see [1]). Since there is a distinguished trivialization on  $Y_\varepsilon$  which extends to  $tN$ , we have the Chern-Simons  $cs_\varepsilon$  defined as a real number. Let  $X_\varepsilon$  be the complement of  $\varepsilon$ -neighborhood of  $\Sigma$  with boundary  $Y_\varepsilon$ . Thus

$$\frac{1}{8\pi^2} \int_{X_\varepsilon} \text{tr} F_A \wedge F_A = k + cs_\varepsilon(A^\alpha).$$

By (2.14) from the reducible connection, we have

$$\lim_{\varepsilon \rightarrow 0} cs_\varepsilon(A^\alpha) = \sum_{i=0}^n \alpha_i \cdot l_i - \frac{1}{2} \left(\sum_{i=0}^n \alpha_i^2\right) \Sigma \cdot \Sigma.$$

So the Chern-Weil formula holds whenever  $A$  is a connection which is smooth and reducible near to  $\Sigma$  by applying the above local statement. Since such connections are dense in  $\mathcal{A}_1^{\alpha,p}$  and the curvature integral is a continuous function of  $A$  in the  $L_{1,A^\alpha}^p$ -topology, the result follows. ■

**Remarks:** (1) When  $\alpha_i = 0$  for all  $i$ , Proposition 2.2 is the usual Chern-Weil formula.

(2) When  $n = 1$ , we have the  $SU(2)$ -situation. Proposition 2.2 for  $\alpha_i = \alpha'_i + 1/2(i = 0, 1)$  and  $l_0 + l_1 = 0$  case is

$$\frac{1}{8\pi^2} \int_{X \setminus \Sigma} \text{tr} F_A \wedge F_A = k + 2(\alpha'_0)l_0 - (\alpha'_0)^2(\Sigma \cdot \Sigma),$$

which is the Proposition 5.7 in [3]. So our formula extends their formula to the  $SU(n+1)$  group.

**Corollary 2.3** *Let  $a$  be the restriction of  $A \in \mathcal{A}_1^{\alpha,p}$  on the boundary of  $X \setminus \Sigma$ . Then the Chern-Simons invariant takes the value*

$$cs(a) \equiv \sum_{i=0}^n \alpha_i l_i - \frac{1}{2} \left( \sum_{i=0}^n \alpha_i^2 \right) \Sigma \cdot \Sigma \pmod{1}. \quad (2.15)$$

■

The proof follows directly from the proof of Proposition 2.2. The Chern-Weil formula gives the charge for singular  $SU(n+1)$ -connections over  $X \setminus \Sigma$ . We study the maximum and minimum of the charge over the conjugacy holonomy region.

**Corollary 2.4** *For  $\Sigma \cdot \Sigma \neq 0$ , the charge takes its maximum and minimum by comparing  $k$  with the following values*

$$k + \frac{\sum_{i=0}^n l_i^2}{2\Sigma \cdot \Sigma} - \frac{j^2 \Sigma \cdot \Sigma}{2(n+1)}, \quad j = 1, 2, \dots, n;$$

$$k + \frac{\sum_{i=0}^{j-1} l_i^2}{2\Sigma \cdot \Sigma} - \frac{m^2 \Sigma \cdot \Sigma}{2j} + \frac{s_j}{j} \left( m - \frac{s_j}{2\Sigma \cdot \Sigma} \right), \quad j = 2, 3, \dots, n; m = 1, \dots, j-1;$$

where  $s_j = \sum_{i=0}^{j-1} l_i$  (not necessary zero).

Proof: First of all, we find out the extreme values inside the region. By the method of Lagrange multiplier,

$$f(\alpha_0, \dots, \alpha_n) = k + \sum_{i=0}^n \alpha_i l_i - \frac{1}{2} \left( \sum_{i=0}^n \alpha_i^2 \right) \Sigma \cdot \Sigma,$$

with constraints  $\sum_{i=0}^n \alpha_i = j$  ( $j = 1, \dots, n$ ), so we have the critical point

$$\alpha_i = \frac{l_i}{\Sigma \cdot \Sigma} + \frac{j}{n+1}, \quad 0 \leq i \leq n,$$

and its corresponding charge is

$$k + \frac{\sum_{i=0}^n l_i^2}{\Sigma \cdot \Sigma} - \frac{j^2 \Sigma \cdot \Sigma}{2(n+1)}, \quad j = 1, \dots, n.$$

For  $j = 0$ , all  $\alpha_i$  are zero, so the charge is  $k$ .

By the method of Lagrange multiplier to locate all critical points of  $f(\alpha_0, \dots, \alpha_n)$  on the boundary  $\alpha_j = \dots = \alpha_n = 0$  ( $j = 2, \dots, n$ ) of the region (2.3), we have the function

$$f(\alpha_0, \dots, \alpha_{j-1}; \lambda_m) = k + \sum_{i=0}^{j-1} \alpha_i l_i - \frac{1}{2} \left( \sum_{i=0}^{j-1} \alpha_i^2 \right) \Sigma \cdot \Sigma - \lambda_m \left( \sum_{i=0}^{j-1} \alpha_i - m \right),$$

with constraints  $\sum_{i=0}^{j-1} \alpha_i = m$  for  $m = 1, \dots, j-1$ . The critical point is

$$\alpha_i = \frac{l_i}{\Sigma \cdot \Sigma} + \frac{m}{j} - \frac{\sum_{i=0}^{j-1} l_i}{j \Sigma \cdot \Sigma}, \quad i = 0, 1, \dots, j-1,$$

and its corresponding charge is, by a straightforward calculation,

$$k + \frac{\sum_{i=0}^{j-1} l_i^2}{2 \Sigma \cdot \Sigma} - \frac{m^2 \Sigma \cdot \Sigma}{2j} + \frac{s_j}{j} \left( m - \frac{s_j}{2 \Sigma \cdot \Sigma} \right),$$

where  $s_j = \sum_{i=0}^{j-1} l_i$ . So the result follows by comparing these extreme values. ■

### 3 Riemann theta function

In this section, the needed number theory criteria are shown. The Riemann theta function enters our picture naturally from the representation of a number as a sum of  $k$ -squares, or by a quadratic form. It is one of well-known classic problems in number theory. Jacobi initially studied this problem by using Riemann theta function as a generating function; in particular, Siegel generalized vastly to several complex variables (see [5]). At the moment we are only interested in the relation between topology and number theory (see §4). Further investigation along this line will be discussed elsewhere.

#### (i) The representation of a number as a sum of nonvanishing squares.

Let  $N$  be an integer,  $N \geq 1$ , with the representation

$$N = N_1^2 + N_2^2 + \dots + N_n^2,$$

where the  $(N_i)_{1 \leq i \leq n}$ 's are integers including zero. Let  $r_n(N)$  denote the number of representations of  $N$  as the sum of  $n$  squares.

1. For  $n = 2$ , Jacobi derived an identity from the generating function

$$\theta_3(0, z) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad q = e^{\pi i z}, \quad \text{with } \text{Im} z > 0.$$

Jacobi identity is

$$\{\theta_3(0, z)\}^2 = 1 + 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{2n-1}}{1 - q^{2n-1}},$$

which gives the result that

$$r_2(N) = 4 \sum_{\text{dodd}, d|N} (-1)^{\frac{1}{2}(d-1)} = 4\{d_1(N) - d_3(N)\}.$$

where  $d_1(N)$  and  $d_3(N)$  are the numbers of the divisors of  $N$  of the form  $4m+1$  and  $4m+3$  respectively.



2. For  $n = 3$ , Legendre proved that a number  $N$  is the sum of three squares if and only if  $N \neq 4^a(8b+7)$ ,  $a \geq 0, b \geq 0$ . For all  $N$ ,  $r_3(4^a N) = r_3(N)$ . The function  $r_3(N)$  has been evaluated by Dirichlet as a finite sum involving symbols of quadratic reciprocity. We give the following formula for  $r_3(N)$  (see [2]):

$$r_3(N) = \frac{G_N}{\pi} \sqrt{N} L(1, \chi),$$

where

$$G_N = \begin{cases} 0 & N \equiv 0, 4, 7 \pmod{8} \\ 16 & N \equiv 3 \pmod{8} \\ 24 & N \equiv 1, 2, 5, 6 \pmod{8} \end{cases}$$

and  $L(s, \chi) = \sum_{m=1}^{\infty} \chi(m) m^{-s}$  with  $\chi(m) = (-4N/m)$ ,  $(r/N)$  the Jacobi symbol.

3. For  $n \geq 4$ , Lagrange (1770) proved that every positive integer can be represented as the sum of four squares, hence also as the sum of five or more squares. In particular for  $n = 4$ ,

$$r_4(N) = 8 \sum_{d|N, 4 \nmid d} d,$$

where the summation is over those positive divisors of  $N$ , which are not divisible by 4.

For our purpose, we need to get the representation  $N$  as the sum of nonvanishing integer squares. Let  $\mathcal{R}_n(N)$  denote the number of representations of  $N$  as the sum of  $n$  nonvanishing squares. For example,  $r_1(3) = r_2(3) = 0$ ,  $r_3(3) = 8$ ,  $r_4(3) = 32$ , but  $\mathcal{R}_4(3) = 0$  (see (3.1)). Note that  $r_1(N) = \mathcal{R}_1(N)$  for all  $N$ . The following proposition gives the relation between  $r_n(N)$  and  $\mathcal{R}_n(N)$ .

**Proposition 3.1** *If  $N$  is an integer ( $N \geq 1$ ), then*

$$\mathcal{R}_n(N) = \sum_{i=1}^n (-1)^{n-i} \binom{n}{i} r_i(N). \quad (3.1)$$

Proof: If  $q = e^{\pi iz}$  with  $\text{Im} z > 0$ , then by definition we have

$$\{\theta_3(0, z)\}^n = \left( \sum_{l=-\infty}^{\infty} q^{l^2} \right)^n = \sum_{N=0}^{\infty} r_n(N) q^N, \quad r_n(0) = 1.$$

The generating function for  $\mathcal{R}_n(N)$  is  $\theta_3(0, z) - 1$ , so

$$\{\theta_3(0, z) - 1\}^n = \left( \sum_{l \neq 0} q^{l^2} \right)^n = \sum_{N=1}^{\infty} \mathcal{R}_n(N) q^N.$$

On the other hand the binomial formula gives

$$\{\theta_3(0, z) - 1\}^n = \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} \{\theta_3(0, z)\}^i.$$

For the constant coefficient, it corresponds to  $i = 0$ , i.e.  $(1 - 1)^n = 0$ . Therefore by comparing the coefficients of  $q^N$  ( $N \geq 1$ ), we have the desired relation. ■

Although we know that for each  $n \geq 5$  all but a finite set of integers are sums of exactly  $n$  nonvanishing squares (see [2] Chapter 6), Proposition 3.1 gives the precise relation among the numbers of representations of  $N$  as sums of squares and sums of nonvanishing squares.

## (ii) The representation of a number by a quadratic form

Let  $(a_{pq})$  be a real, symmetric,  $n \times n$  matrix, and let the associated quadratic form  $Q(x) = \sum_{p,q=1}^n a_{pq}x_px_q$  be positive definite. It is well-known that the multiple series

$$\sum_{i_1, \dots, i_n = -\infty}^{\infty} e^{\pi i z Q(i_1, \dots, i_n)}$$

converges absolutely and uniformly in every compact set in the upper half-plane  $\text{Im} z > 0$ . The theta function associated to  $Q$  is defined to be

$$\theta(z, Q) = \sum_{i_1, \dots, i_n = -\infty}^{\infty} e^{\pi i z Q(i_1, \dots, i_n)}.$$

In case  $a_{pq} = \delta_{pq}$  is the identity matrix, then the  $\theta(z, Id)$  reduces to  $\{\theta_3(0, z)\}^n$ . In our application later, we have the matrix even, i.e.  $a_{pp}$  are even. Then the definition of  $\theta(z, Q)$  yields

$$\theta(z + 1, Q) = \theta(z, Q).$$

In the next section we will consider the particular even matrix:

$$(a_{pq}) = \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{pmatrix}_{n \times n}. \quad (3.2)$$

Its determinant is  $n + 1$ .

**Theorem 3.2** *Let  $(a_{pq})$  be a symmetric,  $n \times n$  matrix of integers, where  $a_{pp}$  are all even for  $p = 1, 2, \dots, n$ , and the associated quadratic form  $Q(x)$  be positive definite with determinant  $D$ . Let  $Q^{-1}$  be the inverse form of  $Q$ . Then we have*

$$\theta(z + 1, Q) = \theta(z, Q), \quad \theta\left(-\frac{1}{z}, Q\right) = \left(\sqrt{\frac{z}{i}}\right)^n D^{-\frac{1}{2}} \theta(z, Q^{-1}),$$

for all complex  $z$  with  $\text{Im} z > 0$ .

From the above relations, one can derive the formula for  $\theta\left(\frac{az+b}{cz+d}, Q\right)$ , with  $a, b, c, d$  are integers and  $ad - bc = 1$ , since the modular group is generated by the two transformations  $A : z \rightarrow z + 1$ , and  $B : z \rightarrow -\frac{1}{z}$  (see [5]).

Let  $r_Q(N)$  (or  $\mathcal{R}_Q(N)$ ) denote the number of (or all nonzero) solutions  $x_1, \dots, x_n$ , with  $x_i$  integral for every  $i$ , such that  $1 \leq i \leq n$  of the equation

$$\sum_{p,q=1}^n a_{pq} x_p x_q = 2N.$$

Let  $(a_{pq})_i$  be the  $(n-1) \times (n-1)$  matrix by deleting  $i$ -th row and  $i$ -th column of the matrix  $(a_{pq})$ . Denote the corresponding quadratic form be  $Q_i$ . Clearly  $Q_i$  is an even, symmetric, positive definite form. Similarly  $Q_{i_1 i_2}$  is the quadratic form with  $x_{i_1} = x_{i_2} = 0$ , etc. The following lemma gives the relation among  $r_Q(N)$ ,  $r_{Q_{i_1 \dots i_j}}(N)$  ( $j = 1, 2, \dots, n-1$ ) and  $\mathcal{R}_Q(N)$ .

**Proposition 3.3** *For the even quadratic form  $Q$ , we have the relation*

$$\begin{aligned} \mathcal{R}_Q(N) = r_Q(N) - \sum_{i=1}^n r_{Q_i}(N) + \\ \sum_{1 \leq i_1 < i_2 \leq n} r_{Q_{i_1 i_2}}(N) - \dots + (-1)^{n-1} \sum_{1 \leq i_1 < \dots < i_{n-1} \leq n} r_{Q_{i_1 \dots i_{n-1}}}(N). \end{aligned} \quad (3.3)$$

Proof: Since  $Q(x)$  is an even form and  $\theta(z, Q)$  is holomorphic for  $\text{Im} z > 0$ , so an expansion of  $\theta(z, Q)$  in power of  $e^{2\pi i z}$  is given by

$$\theta(z, Q) = 1 + \sum_{N=1}^{\infty} r_Q(N) e^{2\pi i N z}, \quad \text{Im} z > 0.$$

There is another way to write the expansion of  $\theta(z, Q)$  as

$$\begin{aligned} \theta(z, Q) &= 1 + \sum_{N=1}^{\infty} \mathcal{R}_Q(N) e^{2\pi i N z} + \sum_{i=1}^n \left( \sum_{x_i=0} e^{\pi i z Q_i(x)} \right) - \sum_{i=1}^n \left( \sum_{x_i=x_j=0} e^{\pi i z Q_i(x)} \right) + \dots \\ &= 1 + \sum_{N=1}^{\infty} \mathcal{R}_Q(N) e^{2\pi i N z} + \\ &\quad \sum_{i=1}^n \left( \sum_{N=1}^{\infty} r_{Q_i}(N) e^{2\pi i N z} \right) - \sum_{i=1}^n \left( \sum_{N=1}^{\infty} r_{Q_{i_1 i_2}}(N) e^{2\pi i N z} \right) + \dots. \end{aligned} \quad (3.4)$$

Hence the relation follows by comparing the coefficients of  $e^{2\pi i N z}$ . ■

In particular, if we take the matrix  $Q = 2Id$ , then

$$r_Q(N) = r_n(N), \quad \text{and} \quad \mathcal{R}_Q(N) = \mathcal{R}_n(N).$$

So the above lemma gives Proposition 3.1. It is clear from our discussion that there are general relations between the number of solutions and the number of nonvanishing solutions via the recursive formula in the theta function.

## 4 Unitary representation of $\pi_1(X \setminus \Sigma)$

In this section, we will use previous results to derive the nontrivial  $SU(n+1)$ -representations of  $\pi_1(X \setminus \Sigma)$ .

**Lemma 4.1** *Let  $A$  be a flat  $\alpha$ -twisted  $SU(n+1)$  connection. Then the holonomy parameter  $(\alpha_i)_{0 \leq i \leq n}$ , the instanton number  $k$  and monopole numbers  $l_i$  are related by*

$$l_i = \alpha_i(\Sigma \cdot \Sigma), \quad \text{for } 0 \leq i \leq n, \quad (4.1)$$

$$k = -\frac{\sum_{i=0}^n l_i^2}{2\Sigma \cdot \Sigma}. \quad (4.2)$$

If  $\Sigma \cdot \Sigma = 0$ , then  $k = 0$  and  $l_i = 0$  for all  $i$ .

Proof: The flat  $\alpha$ -twisted connection  $A$  is one of the model connection corresponding to some integers  $k, l_i$ . Since the bundle is flat, we have

$$w = \text{diag}(c_1(\bar{L}_i) + \alpha_i P.D(\Sigma))_{0 \leq i \leq n},$$

where the 2-form  $w$  is in the proof of Proposition 2.2. Thus for each  $i$ ,  $c_1(\bar{L}_i) + \alpha_i P.D(\Sigma) = 0$ ; the equality  $l_i = \alpha_i(\Sigma \cdot \Sigma)$  follows from integrating over  $\Sigma$ . If  $\Sigma \cdot \Sigma = 0$ ,  $l_i = 0$  for all  $i$  as well.

On the other hand, if  $A$  is flat, the Chern-Weil formula gives

$$\begin{aligned} 0 &= \frac{1}{8\pi^2} \int_{X \setminus \Sigma} \text{tr} F_A \wedge F_A \\ &= k + \sum_{i=0}^n \alpha_i l_i - \frac{1}{2} \left( \sum_{i=0}^n \alpha_i^2 \right) \Sigma \cdot \Sigma \\ &= k + \frac{\sum_{i=0}^n l_i^2}{2\Sigma \cdot \Sigma} \end{aligned}$$

If  $\Sigma \cdot \Sigma = 0$ , we have  $k = 0$  from the second equality. We obtain the formula (4.2).  $\blacksquare$

**Remark:** Note that for  $SU(n+1)$ -flat bundles  $|l_i| < |\Sigma \cdot \Sigma|$  due to  $0 \leq \alpha_i < 1$  from (4.1). All  $\alpha_i$ 's are rational. Using  $\sum_{i=0}^n l_i = 0, l_0 = -\sum_{i=1}^n l_i$  then we have

$$k = -\frac{2 \sum_{i=1}^n l_i^2 + \sum_{i \neq j}^n l_i l_j}{2\Sigma \cdot \Sigma}. \quad (4.3)$$

**Proposition 4.2** *For a simply connected  $X$  and an embedded oriented surface  $\Sigma$  with  $\Sigma \cdot \Sigma \neq 0$ ,*

1. *If  $\sum_{i,j=1}^n l_i l_j = 0$ ,  $\Sigma \cdot \Sigma$  is not a divisor of any  $N(< n(\Sigma \cdot \Sigma)^2)$  with  $\mathcal{R}_n(N) \neq 0$ ;*
2. *In general  $\Sigma \cdot \Sigma$  is not a divisor of any  $N(< \frac{n(n+1)}{2}(\Sigma \cdot \Sigma)^2)$  with  $\mathcal{R}_Q(N) \neq 0$  for  $Q$  as (3.2);*

then  $\pi_1(X \setminus \Sigma)$  has no irreducible representation in  $SU(n+1)$ .

Proof: Suppose there were an irreducible representation  $\rho : \pi_1(X \setminus \Sigma) \rightarrow SU(n+1)$  ( $n \geq 1$ ). The image of  $\rho$  does not contain in any proper subgroup of  $SU(n+1)$ . Denote  $A$  be the corresponding flat connection on  $X \setminus \Sigma$ . By Seifert-Ven Kempf theorem, we have

$$\begin{array}{ccc} \pi_1(Y_\varepsilon) & \rightarrow & \pi_1(X \setminus \Sigma) \\ \downarrow & & \downarrow \\ \pi_1(N_\varepsilon) & \rightarrow & \pi_1(X) = \{1\}. \end{array}$$

So the holonomy on  $\pi_1(X \setminus \Sigma)$  is same as on  $\pi_1(Y_\varepsilon)$ . The space  $Y_\varepsilon$  is the  $S^1$ -bundle over  $\Sigma$ , the homotopy exact sequence of the fibration  $S^1 \rightarrow Y_\varepsilon \rightarrow \Sigma$  yields

$$\{1\} \rightarrow \pi_1(S^1) \rightarrow \pi_1(Y_\varepsilon) \rightarrow \pi_1(\Sigma) \rightarrow \{1\}.$$

In other words,  $\pi_1(Y_\varepsilon)$  is a central extension of  $\pi_1(\Sigma)$ . Let  $\gamma$  be a generator of  $\pi_1(S^1)$ .

Since the conjugacy class  $[\gamma]$  generates  $\pi_1(X \setminus \Sigma)$  and  $\rho$  is an irreducible representation, so the holonomy of  $\rho_A$  is not in  $Z_n$  and other proper subgroups.

Therefore  $\rho_A : \pi_1(X \setminus \Sigma) \rightarrow SU(n+1)/Z_n = PSU(n+1)$ .

$$\pi_1(Y_\varepsilon) = \{a_i, b_i, \gamma | \prod_i [a_i, b_i] = \gamma^m, [\gamma^m, a_i] = 1, [\gamma^m, b_i] = 1\},$$

where  $m = |\Sigma \cdot \Sigma|$  the absolute value of  $\Sigma \cdot \Sigma$ . So the representation  $\rho_A(\gamma)^m = \prod_i [\rho_A(a_i), \rho_A(b_i)]$ , and  $[\rho_A(\gamma)^m, \rho_A(a_i)] = 1, [\rho_A(\gamma)^m, \rho_A(b_i)] = 1$ . This derives that the matrix  $\rho_A(\gamma)$  must be a diagonal matrix:

$$\rho_A(\gamma) = \exp 2\pi i \begin{pmatrix} \alpha_0 & & & \\ & \alpha_1 & & \\ & & \ddots & \\ & & & \alpha_n \end{pmatrix},$$

for  $\alpha_i$ 's in the domain (2.3). Now we can take the flat connection  $A$  as our model connection corresponding to  $\alpha = (\alpha_0, \dots, \alpha_n)$  and instanton number  $k$ , monopole numbers  $l_i$ . By Lemma 4.1 and (4.3), we have

$$\begin{aligned} l_i &= \alpha_i \Sigma \cdot \Sigma \\ k &= -\frac{2 \sum_{i=1}^n l_i^2 + \sum_{i \neq j} l_i l_j}{2 \Sigma \cdot \Sigma} \end{aligned} \quad (4.4)$$

If  $\sum_{i,j=1}^n l_i l_j = 0$ ,  $k = -\frac{\sum_{i=1}^n l_i^2}{\Sigma \cdot \Sigma}$  (an integer), then for (1) (same argument for (2)) the resulting number on the right hand (4.4) is not an integer by the very definition of  $\mathcal{R}_n(N)$  ( $\mathcal{R}_Q(N)$ ) in §3 with  $N = \sum_{i=1}^n l_i^2$  ( $N = \sum_{i=1}^n l_i^2 + \frac{1}{2} \sum_{i,j=1}^n l_i l_j$ ). Note that  $l_i = \alpha_i \Sigma \cdot \Sigma < \Sigma \cdot \Sigma$ , thus the range for  $N = \sum_{i=1}^n l_i^2$  is  $< n(\Sigma \cdot \Sigma)^2$  ( $< \frac{n(n+1)}{2}(\Sigma \cdot \Sigma)^2$ ).

If  $\Sigma \cdot \Sigma = \pm 1$ , then any number  $N < n(\pm 1)^2$  ( $N < \frac{n(n+1)}{2}(\pm 1)^2$ ) has  $\mathcal{R}_n(N) = 0$  ( $\mathcal{R}_Q(N) = 0$ ). So at least one of  $l_i = 0$  for  $N = \sum_{i=1}^n l_i^2$  ( $N = \sum_{i=1}^n l_i^2 + \frac{1}{2} \sum_{i,j=1}^n l_i l_j$ ), i.e. the corresponding  $\alpha_i = 0$ . Hence the induced image of  $\rho_A$  is a proper subgroup

of  $SU(n+1)$ . So  $\pi_1(X \setminus \Sigma)$  has no irreducible  $SU(n+1)$ -representations. ■

### Remarks:

1. We need to use the definition of  $\mathcal{R}_n(N)$  to cover the case of  $\Sigma \cdot \Sigma = \pm 1$ . For  $n = 1$  Proposition 4.2 is the Corollary 5.8 in [3].
2. The condition  $\sum_{i,j=1}^n l_i l_j = 0$  is different from  $\sum_{i,j=0}^n l_i l_j = 0$ . The later one with  $\sum_{i=0}^n l_i = 0$  will imply all  $l_i = 0$ ; we have all  $l_i \neq 0$ , otherwise it will reduce to  $SU(m)$  or  $U(m)$  ( $m < n+1$ ). So we may take  $n+1$  as minimum number of  $l_i \neq 0$ .
3. One can (inductively) apply Proposition 4.2 to  $\mathcal{R}_k(N)$  for non representations of  $\pi_1(X \setminus \Sigma)$  in an rank  $k$  subgroup of  $SU(n+1)$ .

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